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FORCED RESPONSE ANALYSIS FOR MULTI-LAYERED STRUCTURES

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1. INTRODUCTION

The multi-layered structure has been widely used in the fields of mechanical and structural engineering [1, 2]. The primary objective in the analysis of this structure is to understand the lateral vibration of each layer. The transfer matrix method and finite element method have been introduced for numerical analysis in this problem using typical vibration texts [3, 4]. Recently, some analytical methods have been proposed to evaluate the transmissibility of these linear multi-degree-of-freedom systems [5–7]. However, the processes are still complicated when each subsystem of the system is not identical. Moreover, it is more difficult to calculate the analytical dynamic response of the displacement (velocity) and shear force of each layer of the structure using the classical methods.

In this paper, the forced vibration of an N-layer structure will be analyzed by the two-way state-flow graph method [8, 9]. The analytical frequency response of each layer of the structure subjected to a harmonic excitation is evaluated. It is convenient to apply the results of the vibration analysis for periodic excitation. Moreover, computation of the transient response based on the differential equations, which is transferred from the frequency response, for non-periodic excitation is investigated in the article. Finally, the frequency and transient response of a five-story building is examined in numerical examples to illustrate the performance of the method.

2. FREQUENCY RESPONSE ANALYSIS

An N-layer structure subjected to displacement excitation at the end of the system, x_0, as shown in Figure 1 is considered. If the excitation is harmonic, the responses of the displacement and shear force in each layer of the structure are all harmonic with the same frequency. Using the exponential form, the relationship of the shear force and the displacement for the elastic segment in layer i may be expressed as

$$X_i = X_{i-1} + \frac{F_i}{j\omega c_i + k_i}, \tag{1}$$

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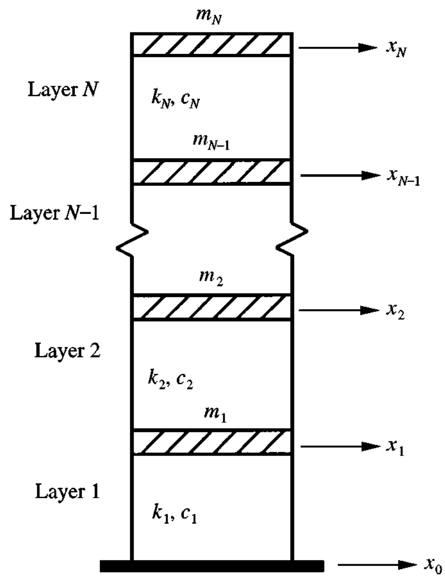

Figure 1. Schematic diagram of a N-layer structure.

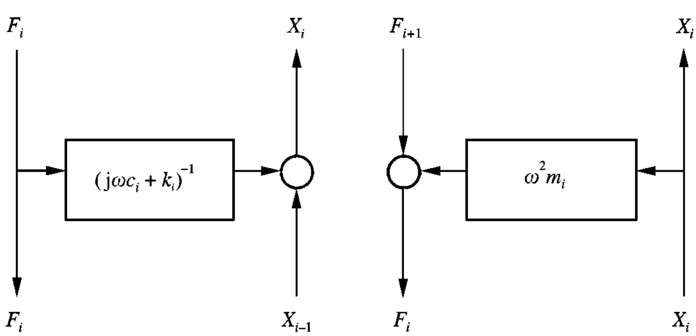

Figure 2. State-flow graph model for the ith layer substructure: (a) elastic segment, (b) lumped mass.

where ω is the excitation frequency, k_i and c_i are the stiffness and damping of the massless elastic segment in layer i. X_i and F_i are the Fourier transform of $x_i(t)$ and $f_i(t)$ in which $x_i(t)$ is the displacement of the lumped mass and $f_i(t)$ is the shear force of the elastic segment in the ith layer of the structure. Based on equation (1), the dynamics of the elastic segment can be expressed as a two-way state-flow graph model as shown in Figure 2(a). For the lumped mass in layer i, the dynamic equation is given as

$$F_i = F_{i+1} + m_i \omega^2 X_i. \tag{2}$$

In the same way, the dynamics of the lumped mass can be described in Figure 2(b). From Figures 2(a) and 2(b), we see that the directions and variables of the state-flow at the higher side of Figure 2(a) match those at the lower side of Figure 2(b).

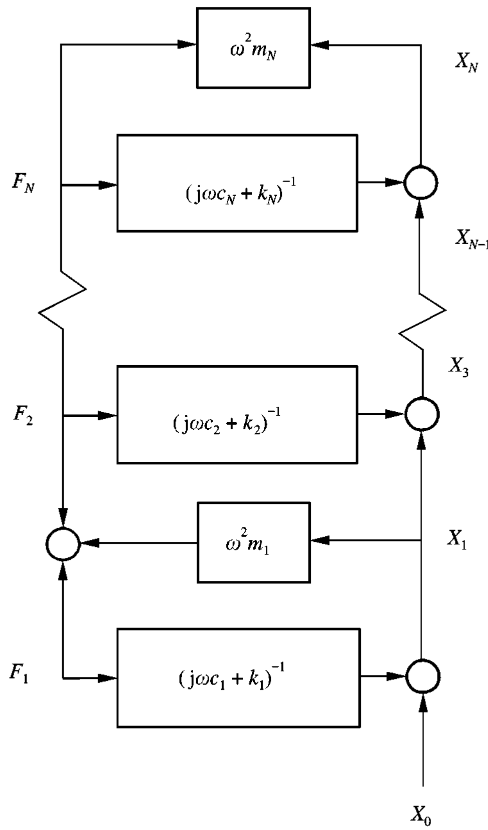

Figure 3. State-flow graph model for the N-layer structure.

Thus, both state-flow models can be connected in a series. This technology can be extended over the entire system to construct the entire state-flow graph model of the dynamic system as shown in Figure 3, in which the input state-flow also matches the direction of the excitation.

Based on the graph model of the entire structure, the frequency response can be calculated using a gain formula [10,8]. From Figure 3, we see that the path of a state-flow forms a closed loop when the state-flow passes through any transverse path from right to left, through a vertical path from a higher to a lower position, through any transverse path from left to right, which is below the previous transverse path, and through the vertical path from a lower to higher position. Thus, there are $N(N-1)/2$ loops in the graph model of the total system. The directions of the state-flow in these loops are all counterclockwise. The loop gain of these loops $L_{i,k}$ is

$$L_{i,k} = \frac{\omega^2 m_k}{j\omega c_i + k_i} \quad \text{for } k = 1, 2, 3, \ldots, N-1, N, \;\; i = 1, 2, 3, \ldots, k-1\ k. \tag{3}$$

When the displacement of the mass in the ith layer is calculated, we see that there is only one forward path from the excitation to the displacement variable X_i. Moreover, the cofactor of this forward path is formed by the part of the graph model, which is higher than X_i. Then, the frequency response of the displacement

X_i, denoted by H_{X_i}, leads to

$$H_{X_i} = \frac{D_i}{D_0}, \tag{4}$$

$$\begin{aligned} D_i = 1 &+ \sum_{i_2=i+1}^{N} \sum_{i_1=i+1}^{i_2} - L_{i_1,i_2} + \sum_{i_4=i+2}^{N} \sum_{i_3=i+2}^{i_4} \sum_{i_2=i+1}^{i_3-1} \sum_{i_1=i+1}^{i_2} \left(\prod_{j=1}^{2} - L_{i_{2j-1},i_{2j}} \right) \\ &+ \sum_{i_{2N-2}=N-1}^{N} \sum_{i_{2N-3}=N-1}^{i_{2N-2}} \sum_{i_{2n-4}=N-2}^{i_{2N-3}-1} \sum_{i_{2N-5}=N-2}^{i_{2N-4}} \cdots \sum_{i_2=i+1}^{i_3-1} \sum_{i_1=i+1}^{i_2} \left(\prod_{j=1}^{N-i-1} - L_{i_{2j-1},i_{2j}} \right) \\ &+ \prod_{j=i+1}^{N-i} - L_{j,j} \end{aligned} \tag{5}$$

If the shear force of the elastic segment in layer i is considered as the output, there are $N - i + 1$ forward paths from the excitation to the output with path gain $\omega^2 m_i$, $\omega^2 m_{i+1}, \ldots$, and $\omega^2 m_N$. The cofactor of this forward path passing through the gain of $\omega^2 m_k$ is formed by the part of the graph model, which is higher than the section of X_k and F_{k+1}. Then, the frequency response of the shear force F_i, H_{F_i}, is given by

$$H_{F_i} = \frac{\omega^2 \sum_{k=i}^{N} m_k D_k}{D_0}. \tag{6}$$

3. TRANSIENT RESPONSE ANALYSIS

For non-periodic excitation, the transient response is usually considered in the vibration analysis. From the frequency responses of the displacement and the shear force as shown in equations (4) and (6), we see that both the equations can be rewritten as a polynomial fractional. Based on the inverse Fourier transformation, both equations of the frequency responses can be transferred to the time domain. Thus, the transient response of the displacement and the shear force may be expressed as the differential equations

$$x_i(t) = \frac{\hat{D}_i}{\hat{D}_0} x_0(t), \tag{7}$$

$$f_i(t) = \frac{- \sum_{k=i}^{N} m_k \hat{D}_k \hat{d}^2}{\hat{D}_0} x_0(t), \tag{8}$$

where

$$\begin{aligned} \hat{D}_i = 1 &+ \sum_{i_2=i+1}^{N} \sum_{i_1=i+1}^{i_2} - \hat{L}_{i_1,i_2} + \sum_{i_4=i+2}^{N} \sum_{i_3=i+2}^{i_4} \sum_{i_2=i+1}^{i_3-1} \sum_{i_1=i+1}^{i_2} \left(\prod_{j=1}^{2} - \hat{L}_{i_{2j-1},i_{2j}} \right) \\ &+ \sum_{i_{2N-2}=N-1}^{N} \sum_{i_{2N-3}=N-1}^{i_{2N-2}} \sum_{i_{2n-4}=N-2}^{i_{2N-3}-1} \sum_{i_{2N-5}=N-2}^{i_{2N-4}} \cdots \sum_{i_2=i+1}^{i_3-1} \sum_{i_1=i+1}^{i_2} \left(\prod_{j=1}^{N-i-1} - \hat{L}_{i_{2j-1},i_{2j}} \right) \\ &+ \prod_{j=i+1}^{N-i} - \hat{L}_{j,j}, \end{aligned} \tag{9}$$

$$\hat{L}_{i,k} = \frac{-m_k\hat{d}^2}{c_1\hat{d} + k_i} \quad \text{for } k = 1, 2, 3, \ldots, N-1, N,\ i = 1, 2, 3, \ldots, k-1, k, \tag{10}$$

with $\hat{d}$ the derivative operator defined as $\hat{d}^i = \mathrm{d}^i/\mathrm{d}t^i$. Since equations (7) and (8) are ordinary differential equations with constant coefficients, many numerical techniques [4, 11, 12] can be applied to calculate the transient response for arbitrary excitation.

4. FORCED VIBRATION OF UNIFORM STRUCTURES

If the mass of each layer is identical, and the damping and stiffness constant of the elastic segment in each layer are also the same, the gain of each closed loop of the graph model will be identical. Replacing each loop gain of equation (3) with the identical loop gain, D_i can be simplified to

$$D_i = \sum_{l=0}^{N-i} \binom{N-i+l}{2l} \left(\frac{-\omega^2 m^*}{\mathrm{j}\omega c^* + k^*} \right)^l, \tag{11}$$

where m^*, c^*, and k^* are the identical mass, damping constant, and the stiffness for each layer of the structure. By substituting equation (11) into equation (4), the frequency response of the displacement in the layer i can be rewritten as the polynomial fraction

$$H_{X_i} = \frac{\sum_{l=0}^{N-i} \binom{N-i+l}{2l} (2\mathrm{j}\xi\Omega\omega + \Omega^2)^{N-l} (-\omega^2)^l}{\sum_{l=0}^{N} \binom{N+l}{2l} (2\mathrm{j}\xi\Omega\omega + \Omega^2)^{N-l} (-\omega^2)^l}, \tag{12}$$

where ξ and Ω are the damping coefficient and natural frequency of each layer of the structure. In the same way, the frequency response of the shear force in the elastic segment of layer i, H_{F_i}, can be obtained by substituting equation (11) into equation (6) leading to

$$H_{F_i} = \frac{m^*\omega^2 \sum_{l=0}^{N-i} \binom{N-i+l+1}{2l+1} (2\mathrm{j}\xi\Omega\omega + \Omega^2)^{N-l} (-\omega^2)^l}{\sum_{l=0}^{N} \binom{N+l}{2l} (2\mathrm{j}\xi\Omega\omega + \Omega^2)^{N-l} (-\omega^2)^l}. \tag{13}$$

In the same way, the differential equations for the calculation of the transient response of the displacement and shear force can be obtained by substituting equation (11) into equations (7) and (8):

$$x_i(t) = \frac{\sum_{l=0}^{N-i} \binom{N-i+l}{2l} (2\xi\Omega\hat{d} + \Omega^2)^{N-l} \hat{d}^{2l}}{\sum_{l=0}^{N} \binom{N+l}{2l} (2\xi\Omega\hat{d} + \Omega^2)^{N-l} \hat{d}^{2l}} x_0(t), \tag{14}$$

$$f_i(t) = \frac{-m^* \sum_{l=0}^{N-i} \binom{N-i+l+1}{2l+1} (2\xi\Omega\hat{d} + \Omega^2)^{N-l} \hat{d}^{2l+1}}{\sum_{l=0}^{N} \binom{N+l}{2l} (2\xi\Omega\hat{d} + \Omega^2)^{N-l} \hat{d}^{2l}} x_0(t). \quad (15)$$

5. NUMERICAL EXAMPLES

To illustrate the application of the present method, a five-story building is considered in the example. The structural properties of every story of the building are the same, namely, mass $m^* = 2{\cdot}0 \times 10^5$ kg, the stiffness coefficient $k^* = 3{\cdot}5 \times 10^8$ N/m, and the damping coefficient $c^* = 2{\cdot}0 \times 10^5$ N s/m. The frequency responses of the displacement and shear force in each story of the building can be directly calculated using equations (12) and (13). The frequency response of the story 3, for example, is given as

$$H_{X_3} = ((-5{\cdot}25 \times 10^3\omega^6 + 5{\cdot}41 \times 10^9\omega^4 - 2{\cdot}82 \times 10^{13}\omega^2 + 1{\cdot}64 \times 10^{16}) + \mathrm{j}(-\omega^7 + 9{\cdot}21 \times 10^6\omega^5 - 6{\cdot}43 \times 10^{10}\omega^3 + 4{\cdot}69 \times 10^{13}\omega))/\hat{D}_0, \quad (16)$$

$$H_{F_3} = \omega^2((1{\cdot}05 \times 10^9\omega^6 - 1{\cdot}09 \times 10^{15}\omega^4 + 7{\cdot}54 \times 10^{18}\omega^2 - 9{\cdot}85 \times 10^{21}) + \mathrm{j}(2 \times 10^5\omega^7 - 1{\cdot}84 \times 10^{12}\omega^5 + 1{\cdot}72 \times 10^{16}\omega^3 - 2{\cdot}81 \times 10^{19}\omega))/\hat{D}_0, \quad (17)$$

where

$$D_0 = (\omega^{10} + 1{\cdot}58 \times 10^4\omega^8 - 5{\cdot}59 \times 10^7\omega^6 + 1{\cdot}88 \times 10^{11}\omega^4 - 1{\cdot}41 \times 10^{14}\omega^2 + 1.64 \times 10^{16}) + \mathrm{j}(9\omega^9 - 9{\cdot}81 \times 10^4\omega^7 + 3{\cdot}22 \times 10^8\omega^5 - 3{\cdot}22 \times 10^{11}\omega^3 + 4.69 \times 10^{13}\omega). \quad (18)$$

If a non-periodic excitation is considered as

$$x_0(t) = \begin{cases} 1 - \cos(\Omega_n t) & \text{for } 0 \geqslant t \geqslant 2\pi/\Omega_n \\ 0 & \text{for otherwise.} \end{cases} \quad (19)$$

with Ω_n being the natural frequency of each layer of the building. The transient response of the displacement and shear force can be calculated using the differential equations, which are obtained from the equations of the frequency response by simply replacing $\mathrm{j}\omega$ with the derivative operator $\hat{d}$. The transient response of stories 1, 3, and 5 are shown in Figures 4 and 5.

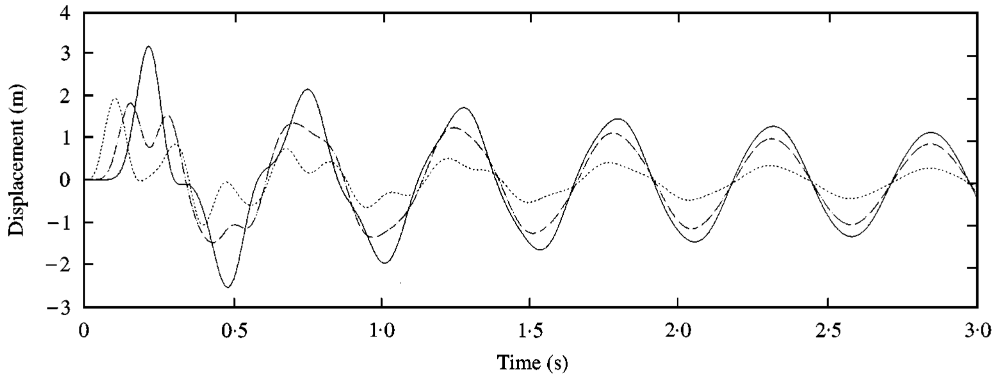

Figure 4. Transient response of the displacement of stories 1, 3, and 5. (. . . . .) 1st story, (- - - -) 3rd story, (——) 5th story.

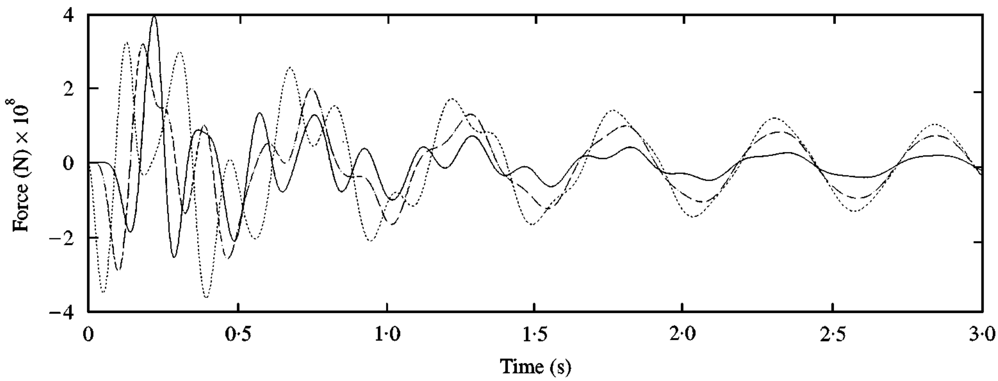

Figure 5. Transient response of the shear force of stories 1, 3, and 5. (. . . . .) 1st story, (- - - -) 3rd story, (——) 5th story.

6. CONCLUSIONS

The frequency responses of the shear force and lateral displacement of multi-layered structures subjected to an excitation from one end of the structure has been evaluated using an analytical method. In the investigation, the multi-layered structure is represented by a two-way graph model, from which the frequency response of each layer can be directly calculated. Moreover, succinct forms for representing the response of uniform-layered structures are also derived. Since the results are represented analytically, the computation error will be reduced to a minimum. Based on the derived frequency responses, the differential equations for calculating the transient response of arbitrary excitation is easily obtained. Finally, the frequency and the transient response of a five-story building subjected to a periodic and a non-periodic load, respectively, have been investigated in the numerical example to show the efficiency of this method in a realistic implementation.

REFERENCES

1. J. N. YANG, Y. K. LIN and S. SAE-UNG 1980 *Journal of Engineering Mechanics Division ASCE* **106**, 801–817. Tall building response to earthquake excitations.
2. R. E. D. BISHOP, W. G. PRICE and P. K. Y. TAM 1977 *International Shipbuilding Progress.* **24**, 284–295. Wave-induced response of a flexible ship.
3. W. T. THOMSON 1993 *Theory of Vibration with Applications*. Englewood Cliffs, NJ: Prentice-Hall, Fourth edition.
4. R. W. CLOUGH and J. PENZIEN 1993 *Dynamics of Structures*. New York: McGraw-Hill, Second edition.
5. M. L. MUNJAL, A. V. SREENATH and M. V. NARASIMHAN 1973 *Journal of Sound and Vibration* **26**, 193–208. An algebraic algorithm for the design and analysis of linear dynamical systems.
6. S. A. PAIPETIS and A. F. VAKAKIS 1985 *Journal of Sound and Vibration* **98**, 13–23. A method for unidirectional vibration isolators with many degrees of freedom.
7. A. F. VAKAKIS and S. A. PAIPETIS 1985 *Journal of Sound and Vibration* **99**, 557–562. Transient response of unidirectional vibration isolators with many degrees of freedom.
8. W. J. HSUEH 1998 *Journal of Sound and Vibration* **216**, 399–412. Analysis of vibration isolation systems using a graph model.
9. W. J. HSUEH 1999 *Journal of Sound and Vibrations* **224**, 209–220. On the vibration analysis of multi-branch torsional systems.
10. S. J. MASON 1956 *Proceedings of IRE* **44**, 920–926. Feedback theory—further properties of signal flow graph.
11. R. W. HAMMING 1962 *Numerical Methods for Scientists and Engineers*. New York: McGraw-Hill.
12. J. R. RICE 1983 *Numerical Methods, Software, and Analysis*. New York: McGraw-Hill.